Name (Please Print) ______________________

TAS/TDS- Back paper Exam - **Maximum Time: 3 hours** **Maximum Score: 100**

1. (40 points) Let $\mathcal{A}_n$ be the events that are observable by time n.Let $N \in \mathbb{N}$. Consider

$$\Omega_N = \{\omega = (\omega_1, \omega_2, \ldots, \omega_N) : \omega_i \in \{-1, +1\}$$

equipped with the uniform distribution, denoted by $\mathbb{P} \equiv \mathbb{P}_N$. For $1 \leq k \leq N$, let $X_K : \Omega_N \to \{-1, 1\}$ be given by $X_k(\omega) = \omega_k$ and for $1 \leq n \leq N$, let $S_n : \Omega_N \to \{-1, 1\}$ be given by $S_n(\omega) = \sum_{k=1}^{n} X_k(\omega)$ and $S_0 = 0$.

   (a) Show that $\mathcal{A}_n$ is closed under complimentation and intersections.

   (b) For $1 \leq n \leq N$, show that the mode of S_n is $\{0, 1\}$ that is

$$\max\{\mathbb{P}(S_n = a) : a \in \mathbb{Z}\} = \begin{cases} \mathbb{P}(S_{2k} = 0) & \text{if } n = 2k, k \in \mathbb{N} \\ \mathbb{P}(S_{2k-1} = 1) & \text{if } n = 2k-1, k \in \mathbb{N} \end{cases} = \binom{2k}{k}\frac{1}{2^{2k}}$$

   (c) For $a < b, a, b \in \mathbb{Z}$, $1 \leq n \leq N$ show that

$$\mathbb{P}(a \leq S_n \leq b) \leq (b - a + 1)\mathbb{P}(S_n \in \{0, 1\})$$

   and conclude that $\lim\limits_{N \to \infty} \mathbb{P}(a \leq S_N \leq b) = 0$.

   (d) Let $-\infty < a < 0 < b < \infty, a, b \in \mathbb{Z},,$

$$\sigma_a = \min\{k \geq 1 : S_k = a\} \qquad \text{and} \qquad \sigma_b = \min\{k \geq 1 : S_k = b\}.$$

   (e) Let $a \in \mathbb{N}$ and $\sigma_a = \min\{k \geq 1 : S_k = a\}$. Show that

$$\mathbb{P}(\sigma_a = n) = \frac{a}{n}\mathbb{P}(S_n = a)$$

2. (20 points) For $x \in \mathbb{Z}^d$, let $| x |= \sum_{i=1}^{d} | x_i |$. Let S_n be the simple symmetric walk on $\mathbb{Z}^d$. Let

$$\tau_R = \inf\{n \geq 0 :| S_n |= R\}.$$

Let $h : \mathbb{Z}^d \to [0, \infty)$ be given by

$$h(x) = \mathbb{P}_x(\tau_{30} < \tau_1).$$

Show that

   (a) $h(x) = 1$ whenever $| x |\geq 30$

   (b) $h(x) = 0$ whenever $| x |\leq 1$

   (c) h is harmonic on the set $1 <| x |< 30$, i.e.

$$h(x) = \frac{1}{2d}\left(\sum_{i=1}^{d} h(x + e_d) + h(x - e_d)\right),$$

   whenever $1 <| x |< 30$, where $\{e_i : 1 \leq i \leq d\}$ are the standard basis for $\mathbb{Z}^d$.

3. (20 points) Assume the following version of:

Cramer's Theorem: Let (X_i) be i.i.d. $\mathbb{R}$–valued random variables such that

$$0 \in \text{interior}\{t \in \mathbb{R} : \varphi(t) = \mathbb{E}\, e^{tX_1} < \infty\} \tag{1}$$

Let $S_n = \sum_{i=1}^n X_i$. Then for all $a > \mathbb{E}X_1$

$$\lim_{n\to\infty} \frac{1}{n} \log \mathbb{P}(S_n \geq an) = -I(a), \tag{2}$$

where

$$I(z) = \sup_{t\in\mathbb{R}} [zt - \log \varphi(t)].$$

Find I: when $X_i \sim$

(a) X with $\mathbb{P}(X = a) = 1$ for some $a \in \mathbb{R}$.

(b) X where $\mathbb{P}(X = 1) = \mathbb{P}(X = 2) = \mathbb{P}(X = 3) = \frac{1}{3}$.

4. (20 points) Consider a martingale where Z_n can take on only the values 4^{-n-1} and $1 - 4^{-n-1}$, each with probability $\frac{1}{2}$.

(a) Given that Z_n, conditional on Z_{n-1}, is independent of $Z_{n-2}, Z_{n-3}, \ldots, Z_1$ find $E[Z_n \mid Z_{n-1}]$ for each n so that the martingale condition is satisfied.

(b) Show that $\mathbb{P}(\sup_{n\geq 1} Z_n \geq 1) = \dfrac{1}{2} \neq 0 = \mathbb{P}(\bigcup_{n\geq 1} \{Z_n \geq 1\})$

(c) Show that for all $\epsilon > 0$, $\mathbb{P}(\sup_{n\geq 1} Z_n \geq a) \leq \dfrac{\mathbb{E}[Z_1]}{a - \epsilon}$.